# Mixed models in R using the lme4 package
# Part 3: Longitudinal data

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## Outline

## Simple longitudinal data

- *Repeated measures* data consist of measurements of a response (and, perhaps, some covariates) on several *experimental* (or observational) *units*.
- Frequently the experimental (observational) unit is `Subject` and we will refer to these units as "subjects". However, the methods described here are not restricted to data on human subjects.
- *Longitudinal* data are repeated measures data in which the observations are taken over time.
- We wish to characterize the response over time within subjects and the variation in the time trends between subjects.
- Frequently we are not as interested in comparing the particular subjects in the study as much as we are interested in modeling the variability in the population from which the subjects were chosen.

## Sleep deprivation data

- This laboratory experiment measured the effect of sleep deprivation on cognitive performance.
- There were 18 subjects, chosen from the population of interest (long-distance truck drivers), in the 10 day trial. These subjects were restricted to 3 hours sleep per night during the trial.
- On each day of the trial each subject's reaction time was measured. The reaction time shown here is the average of several measurements.
- These data are *balanced* in that each subject is measured the same number of times and on the same occasions.

## Reaction time versus days by subject
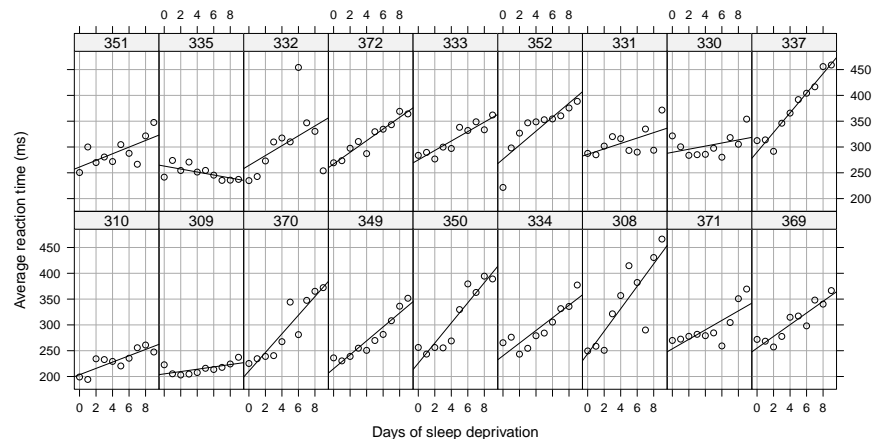


## Comments on the sleep data plot

- ▶ The plot is a "trellis" or "lattice" plot where the data for each subject are presented in a separate panel. The axes are consistent across panels so we may compare patterns across subjects.
- ▶ A reference line fit by simple linear regression to the panel's data has been added to each panel.
- ▶ The aspect ratio of the panels has been adjusted so that a typical reference line lies about $45°$ on the page. We have the greatest sensitivity in checking for differences in slopes when the lines are near $\pm 45°$ on the page.
- ▶ The panels have been ordered not by subject number (which is essentially a random order) but according to increasing intercept for the simple linear regression. If the slopes and the intercepts are highly correlated we should see a pattern across the panels in the slopes.

## Assessing the linear fits

- ▶ In most cases a simple linear regression provides an adequate fit to the within-subject data.
- ▶ Patterns for some subjects (e.g. 350, 352 and 371) deviate from linearity but the deviations are neither widespread nor consistent in form.
- ▶ There is considerable variation in the intercept (estimated reaction time without sleep deprivation) across subjects – 200 ms. up to 300 ms. – and in the slope (increase in reaction time per day of sleep deprivation) – 0 ms./day up to 20 ms./day.
- ▶ We can examine this variation further by plotting confidence intervals for these intercepts and slopes. Because we use a pooled variance estimate and have balanced data, the intervals have identical widths.
- ▶ We again order the subjects by increasing intercept so we can check for relationships between slopes and intercepts.
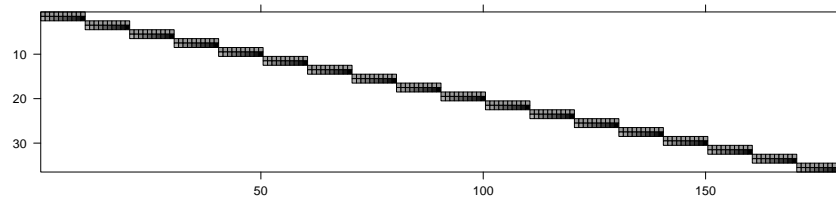
## 95% conf int on within-subject intercept and slope
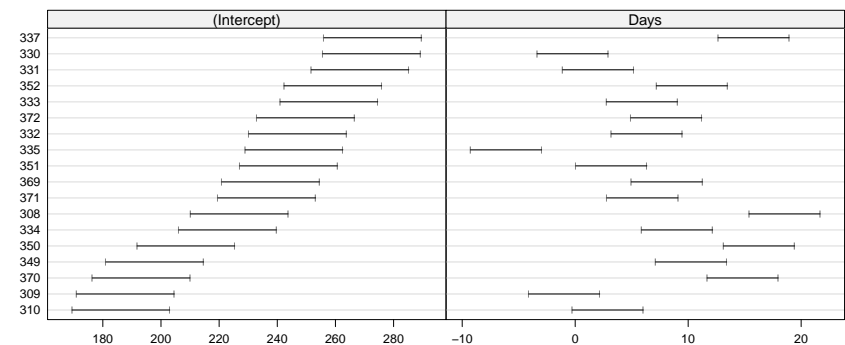


These intervals reinforce our earlier impressions of considerable variability between subjects in both intercept and slope but little evidence of a relationship between intercept and slope.

## A preliminary mixed-effects model

- We begin with a linear mixed model in which the fixed effects $[\beta_1, \beta_2]'$ are the representative intercept and slope for the population and the random effects $\mathbf{b}_i = [b_{i1}, b_{i2}]', i = 1, \ldots, 18$ are the deviations in intercept and slope associated with subject $i$.

- The random effects vector, $\mathbf{b}$, consists of the $18$ intercept effects followed by the $18$ slope effects.



## Terms and matrices

- The term `Days` in the formula generates a model matrix $\mathbf{X}$ with two columns, the intercept column and the numeric `Days` column. (The intercept is included unless suppressed.)

- The term `(Days|Subject)` generates a vector-valued random effect (intercept and slope) for each of the $18$ levels of the `Subject` factor.

## Fitting the model

```
> (fm1 <- lmer(Reaction ~ Days + (Days | Subject),
+      sleepstudy))

Linear mixed model fit by REML ['merMod']
Formula: Reaction ~ Days + (Days | Subject)
   Data: sleepstudy
REML criterion at convergence: 1743.628
Random effects:
 Groups    Name         Variance Std.Dev. Corr
 Subject   (Intercept)  612.09   24.740
           Days          35.07    5.922   0.066
 Residual               654.94   25.592
Number of obs: 180, groups: Subject, 18

Fixed effects:
            Estimate Std. Error t value
(Intercept)  251.405      6.825   36.84
Days          10.467      1.546    6.77

Correlation of Fixed Effects:
     (Intr)
Days -0.138
```

## A model with uncorrelated random effects

- The data plots gave little indication of a systematic relationship between a subject's random effect for slope and his/her random effect for the intercept. Also, the estimated correlation is quite small.

- We should consider a model with uncorrelated random effects. To express this we use two random-effects terms with the same grouping factor and different left-hand sides. In the formula for an `lmer` model, distinct random effects terms are modeled as being independent. Thus we specify the model with two distinct random effects terms, each of which has `Subject` as the grouping factor. The model matrix for one term is intercept only (`1`) and for the other term is the column for `Days` only, which can be written `0+Days`. (The expression `Days` generates a column for `Days` and an intercept. To suppress the intercept we add `0+` to the expression; `-1` also works.)

## A mixed-effects model with independent random effects

```
Linear mixed model fit by REML ['merMod']
Formula: Reaction ~ Days + (1 | Subject) + (0 + Days | Subject)
   Data: sleepstudy
REML criterion at convergence: 1743.669
Random effects:
 Groups    Name        Variance Std.Dev.
 Subject   (Intercept) 627.57   25.051
 Subject   Days         35.86    5.988
 Residual              653.58   25.565
Number of obs: 180, groups: Subject, 18

Fixed effects:
            Estimate Std. Error t value
(Intercept)  251.405      6.885   36.51
Days          10.467      1.560    6.71

Correlation of Fixed Effects:
     (Intr)
Days -0.184
```

## Comparing the models

- Model `fm1` contains model `fm2` in the sense that if the parameter values for model `fm1` were constrained so as to force the correlation, and hence the covariance, to be zero, and the model were re-fit, we would get model `fm2`.
- The value `0`, to which the correlation is constrained, is not on the boundary of the allowable parameter values.
- In these circumstances a likelihood ratio test and a reference distribution of a $\chi^2$ on 1 degree of freedom is suitable.

```
> anova(fm2, fm1)

Data: sleepstudy
Models:
fm2: Reaction ~ Days + (1 | Subject) + (0 + Days | Subject)
fm1: Reaction ~ Days + (Days | Subject)
    Df    AIC    BIC  logLik deviance  Chisq Chi Df Pr(>Chisq)
fm2  5 1762.0 1778.0 -876.00   1752.0
fm1  6 1763.9 1783.1 -875.97   1751.9 0.0639      1     0.8004
```

## Conclusions from the likelihood ratio test

- Because the large p-value indicates that we would not reject `fm2` in favor of `fm1`, we prefer the more parsimonious `fm2`.
- This conclusion is consistent with the AIC (Akaike's Information Criterion) and the BIC (Bayesian Information Criterion) values for which "smaller is better".
- We can also use a Bayesian approach, where we regard the parameters as themselves being random variables, is assessing the values of such parameters. A currently popular Bayesian method is to use sequential sampling from the conditional distribution of subsets of the parameters, given the data and the values of the other parameters. The general technique is called *Markov chain Monte Carlo* sampling.
- We will expand on the use of likelihood-ratio tests in the next section.

## Conditional means of the random effects

```
> (rr2 <- ranef(fm2))

$Subject
    (Intercept)        Days
308   1.5126973   9.3234890
309 -40.3738973  -8.5991692
310 -39.1810413  -5.3877906
330  24.5189035  -4.9686455
331  22.9144326  -3.1939346
332   9.2219735  -0.3084935
333  17.1561209  -0.2872072
334  -7.4517332   1.1159900
335   0.5787240 -10.9059660
337  34.7679284   8.6276161
349 -25.7543231   1.2806876
350 -13.8650342   6.7564002
351   4.9159797  -3.0751328
352  20.9290426   3.5122097
369   3.2586474   0.8730507
370 -26.4758256   4.9837861
371   0.9056474  -1.0052929
372  12.4217574   1.2584029
attr(,"class")
```

## Scatterplot of the conditional means
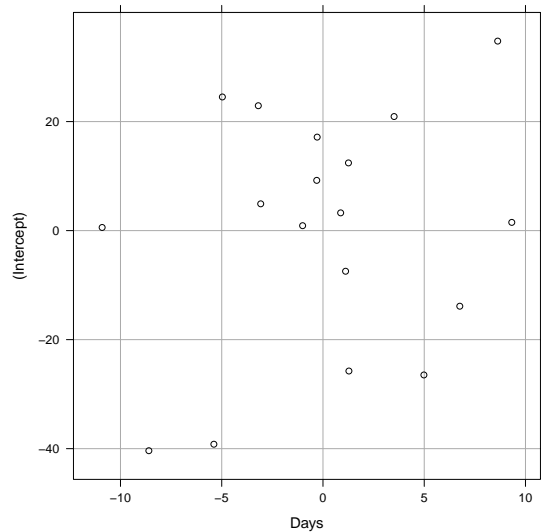


## Comparing within-subject coefficients

- ▶ For this model we can combine the conditional means of the random effects and the estimates of the fixed effects to get conditional means of the within-subject coefficients.
- ▶ These conditional means will be "shrunken" towards the fixed-effects estimates relative to the estimated coefficients from each subject's data. John Tukey called this "borrowing strength" between subjects.
- ▶ Plotting the shrinkage of the within-subject coefficients shows that some of the coefficients are considerably shrunken toward the fixed-effects estimates.
- ▶ However, comparing the within-group and mixed model fitted lines shows that large changes in coefficients occur in the noisy data. Precisely estimated within-group coefficients are not changed substantially.
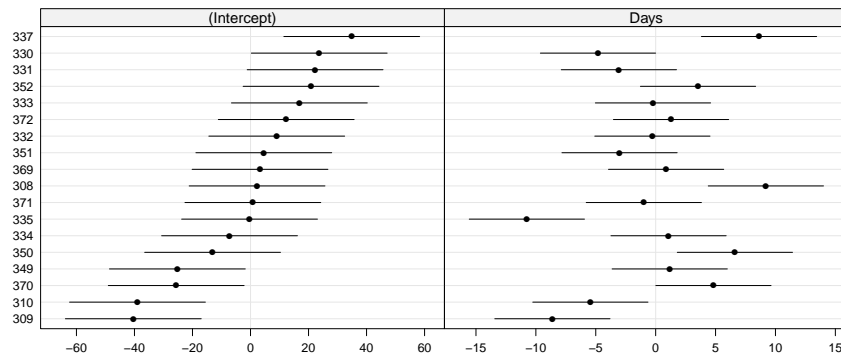
## Estimated within-group coefficients and BLUPs
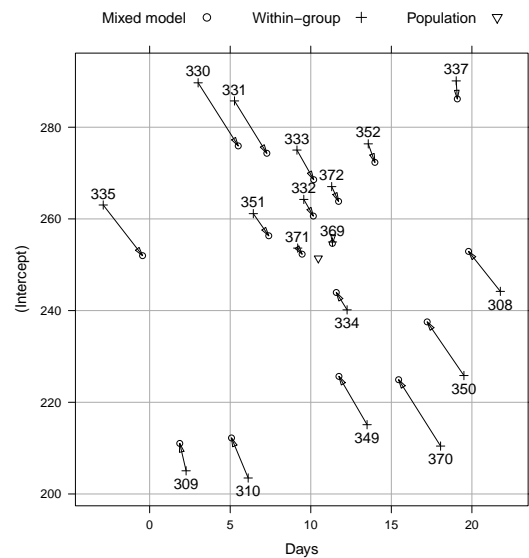


## Observed and fitted
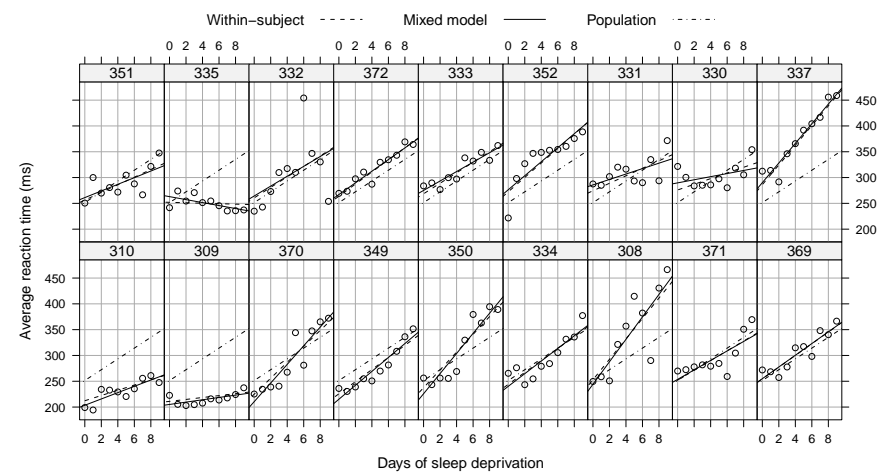
## Plot of prediction intervals for the random effects



Each set of prediction intervals have constant width because of the balance in the experiment.

## Conclusions from the example

- ▶ Carefully plotting the data is enormously helpful in formulating the model.
- ▶ It is relatively easy to fit and evaluate models to data like these, from a balanced designed experiment.
- ▶ We consider two models with random effects for the slope and the intercept of the response w.r.t. time by subject. The models differ in whether the (marginal) correlation of the vector of random effects per subject is allowed to be nonzero.
- ▶ The "estimates" (actually, the conditional means) of the random effects can be considered as penalized estimates of these parameters in that they are shrunk towards the origin.
- ▶ Most of the prediction intervals for the random effects overlap zero.

## Random slopes and interactions

- ▶ In the `sleepstudy` model fits we allowed for random effects for `Days` by `Subject`.
- ▶ These random effects can be considered as an interaction between the fixed-effects covariate `Days` and the random-effects factor `Subject`.
- ▶ When we have both fixed-levels categorical covariates and random-levels categorical covariates we have many different ways in which interactions can be expressed.
- ▶ Often the wide range of options provides "enough rope to hang yourself" in the sense that it is very easy to create an overly-complex model.

## The `Multilocation` data set

- ▶ Data from a multi-location trial of several treatments are described in section 2.8 of Littell, Milliken, Stroup and Wolfinger (1996) **SAS System for Mixed Models** and are available as `Multilocation` in package `SASmixed`.
- ▶ Littell et al. don't cite the source of the data. Apparently `Adj` is an adjusted response of some sort for 4 different treatments applied at each of 3 blocks in each of 9 locations. Because `Block` is implicitly nested in `Location`, the `Grp` interaction variable was created.

```
> str(Multilocation)

'data.frame': 108 obs. of  7 variables:
 $ obs     : num  3 4 6 7 9 10 12 16 19 20 ...
 $ Location: Factor w/ 9 levels "A","B","C","D",..: 1 1 1 1 1 1..
 $ Block   : Factor w/ 3 levels "1","2","3": 1 1 1 1 2 2 2 2 3 ..
 $ Trt     : Factor w/ 4 levels "1","2","3","4": 3 4 2 1 2 1 3 ..
 $ Adj     : num  3.16 3.12 3.16 3.25 2.71 ...
 $ Fe      : num  7.1 6.68 6.83 6.53 8.25 ...
 $ Grp     : Factor w/ 27 levels "A/1","A/2","A/3",..: 1 1 1 1 ..
```
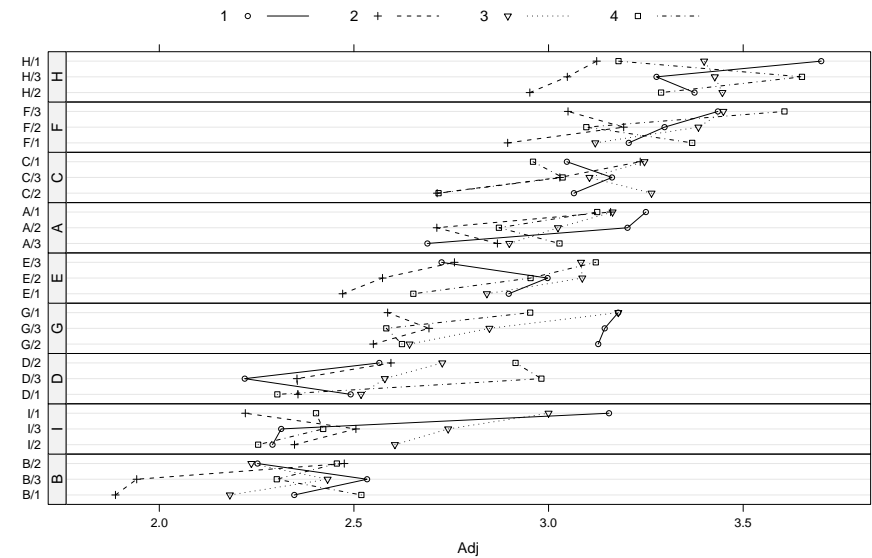
## Response by `Grp` and `Trt`
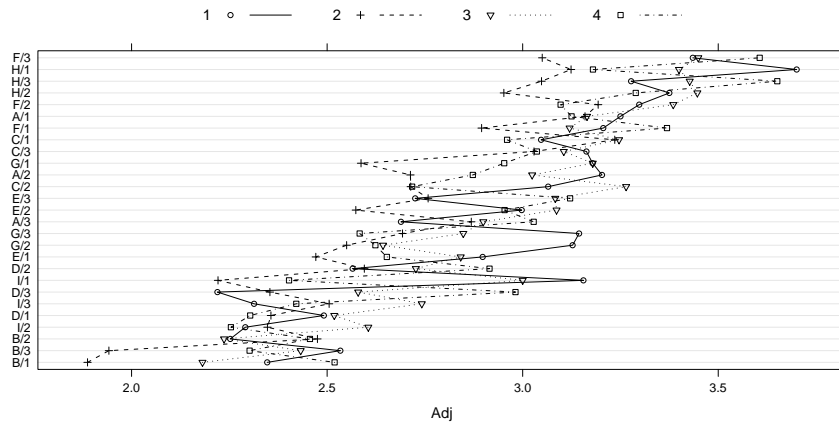


▶ From this one plot (Littell et al. do not provide any plots but instead immediately jump into fitting several "cookie-cutter" models) we see that there are differences between locations, not as much between blocks within location, and that treatment 2 is providing a lower adjusted response.

## Response by `Block` and `Trt` within `Location`



## Fixed-levels categorical covariates and "contrasts"

▶ In this experiment we are interested in comparing the effectiveness of these four levels of `Trt`.

▶ That is, the levels of `Trt` are fixed levels and we should incorporate them in the fixed-effects part of the model.

▶ Unlike the situation with random effects, we cannot separately estimate "effects" for each level of a categorical covariate in the fixed-effects and an overall intercept term.

▶ We could suppress the intercept term but even then we still encounter redundancies in effects for each level when we have more than one categorical covariate in the fixed-effects.

▶ Because of this we estimate coefficients for $k-1$ "contrasts" associated with the $k$ levels of a factor.

▶ The default contrasts (called `contr.treatment`) measure changes relative to a reference level which is the first level of the factor. Other contrasts can be used when particular comparisons are of interest.

## A simple model for `Trt` controlling for `Grp`

```
> print(fm3 <- lmer(Adj ~ Trt + (1 | Grp), Multilocation),
+     corr = FALSE)

Linear mixed model fit by REML ['merMod']
Formula: Adj ~ Trt + (1 | Grp)
   Data: Multilocation
REML criterion at convergence: 31.5057
Random effects:
 Groups    Name        Variance Std.Dev.
 Grp      (Intercept) 0.11092  0.3331
 Residual             0.03672  0.1916
Number of obs: 108, groups: Grp, 27

Fixed effects:
            Estimate Std. Error t value
(Intercept)  2.92401    0.07395   39.54
Trt2        -0.24637    0.05215   -4.72
Trt3         0.02544    0.05215    0.49
Trt4        -0.05834    0.05215   -1.12
```

## Interpretation of the results

- We see that the variability between the Location/Block combinations (levels of `Grp`) is greater than the residual variability, indicating the importance of controlling for it.
- The contrast between levels 2 and 1 of `Trt`, labeled `Trt2` is the greatest difference and apparently significant.
- If we wish to evaluate the "significance" of the levels of `Trt` as a group, however, we should fit the trivial model and perform a LRT.

```
> fm4 <- lmer(Adj ~ 1 + (1 | Grp), Multilocation)
> anova(fm4, fm3)

Data: Multilocation
Models:
fm4: Adj ~ 1 + (1 | Grp)
fm3: Adj ~ Trt + (1 | Grp)
    Df    AIC    BIC  logLik deviance Chisq Chi Df Pr(>Chisq)
fm4  3 49.731 57.777 -21.8654   43.731
fm3  6 26.951 43.044  -7.4756   14.951 28.78      3  2.491e-06
```

## An interaction between fixed-effects factors

- We could ask if there is an interaction between the levels of `Trt` and those of `Location` considered as fixed effects.

```
> anova(fm6 <- lmer(Adj ~ Location * Trt + (1 | Grp),
+        Multilocation))

Analysis of Variance Table
             Df Sum Sq Mean Sq F value
Location      8 6.9475 0.86843 25.1147
Trt           3 1.2217 0.40725 11.7774
Location:Trt 24 0.9966 0.04152  1.2008

> anova(fm5, fm6)

Data: Multilocation
Models:
fm5: Adj ~ Location + Trt + (1 | Grp)
fm6: Adj ~ Location * Trt + (1 | Grp)
    Df     AIC    BIC logLik deviance  Chisq Chi Df Pr(>Chisq)
fm5 14 -24.504 13.046 26.252  -52.504
fm6 38 -11.146 90.775 43.573  -87.146 34.642     24    0.07388
```

## `Location` as a fixed-effect

- We have seen that `Location` has a substantial effect on `Adj`. If we are interested in these specific 9 locations we could incorporate them as fixed-effects parameters.
- Instead of examining 8 coefficients separately we will consider their cumulative effect using the single-argument form of `anova`.

```
> anova(fm5 <- lmer(Adj ~ Location + Trt + (1 | Grp),
+        Multilocation))

Analysis of Variance Table
         Df Sum Sq Mean Sq F value
Location  8 7.3768 0.92210  25.115
Trt       3 1.2217 0.40725  11.092
```

## Considering levels of `Location` as random effects

```
> print(fm7 <- lmer(Adj ~ Trt + (1 | Location) + (1 |
+        Grp), Multilocation), corr = FALSE)

Linear mixed model fit by REML ['merMod']
Formula: Adj ~ Trt + (1 | Location) + (1 | Grp)
   Data: Multilocation
REML criterion at convergence: 1.8978
Random effects:
 Groups   Name        Variance Std.Dev.
 Grp      (Intercept) 0.005085 0.07131
 Location (Intercept) 0.114657 0.33861
 Residual             0.036715 0.19161
Number of obs: 108, groups: Grp, 27; Location, 9

Fixed effects:
            Estimate Std. Error t value
(Intercept)  2.92401    0.11953  24.462
Trt2        -0.24637    0.05215  -4.724
Trt3         0.02544    0.05215   0.488
Trt4        -0.05834    0.05215  -1.119
```

## Is `Grp` needed in addition to `Location`?

- ▶ At this point we may want to check whether the random effect for `Block` within `Location` is needed in addition to the random effect for `Location`.

```
> fm8 <- lmer(Adj ~ Trt + (1 | Location), Multilocation)
> anova(fm8, fm7)
Data: Multilocation
Models:
fm8: Adj ~ Trt + (1 | Location)
fm7: Adj ~ Trt + (1 | Location) + (1 | Grp)
    Df     AIC    BIC logLik deviance  Chisq Chi Df Pr(>Chisq)
fm8  6 0.25442 16.347 5.8728  -11.746
fm7  7 0.39496 19.170 6.8025  -13.605 1.8595      1     0.1727
```

- ▶ Apparently not, but we may want to revisit this issue after checking for interactions.

## Ways of modeling random/fixed interactions

- ▶ There are two ways we can model the interaction between a fixed-levels factor (`Trt`) and a random-levels factor (`Location`, as we are currently viewing this factor).
- ▶ The first, and generally preferable, way is to incorporate a simple scalar random-effects term with the interaction as the grouping factor.
- ▶ The second, more complex, way is to use vector-valued random effects for the random-levels factor. We must be careful when using this approach because it often produces a degenerate model, but not always obviously degenerate.

## Scalar random effects for interaction

```
> (fm9 <- lmer(Adj ~ Trt + (1 | Trt:Location) + (1 |
+      Location), Multilocation, REML = FALSE))
Linear mixed model fit by maximum likelihood ['merMod']
Formula: Adj ~ Trt + (1 | Trt:Location) + (1 | Location)
   Data: Multilocation
     AIC      BIC   logLik deviance
  2.2544  21.0293   5.8728 -11.7456
Random effects:
 Groups       Name        Variance Std.Dev.
 Trt:Location (Intercept) 0.0000   0.0000
 Location     (Intercept) 0.1029   0.3207
 Residual                 0.0393   0.1982
Number of obs: 108, groups: Trt:Location, 36; Location, 9

Fixed effects:
            Estimate Std. Error t value
(Intercept)  2.92401    0.11351  25.759
Trt2        -0.24637    0.05396  -4.566
Trt3         0.02544    0.05396   0.472
Trt4        -0.05834    0.05396  -1.081

Correlation of Fixed Effects:
```

## Both interaction and Block-level random effects

```
> (fm10 <- update(fm9, . ~ . + (1 | Grp)))
Linear mixed model fit by maximum likelihood ['merMod']
Formula: Adj ~ Trt + (1 | Trt:Location) + (1 | Location) + (1 | Grp)
   Data: Multilocation
     AIC      BIC   logLik deviance
  2.3564  23.8134   6.8218 -13.6436
Random effects:
 Groups       Name        Variance  Std.Dev.
 Trt:Location (Intercept) 0.0007769 0.02787
 Grp          (Intercept) 0.0056193 0.07496
 Location     (Intercept) 0.1011949 0.31811
 Residual                 0.0345787 0.18595
Number of obs: 108, groups: Trt:Location, 36; Grp, 27; Location, 9

Fixed effects:
            Estimate Std. Error t value
(Intercept)  2.92401    0.11322  25.826
Trt2        -0.24637    0.05229  -4.712
Trt3         0.02544    0.05229   0.487
Trt4        -0.05834    0.05229  -1.116

Correlation of Fixed Effects:
```

## Scalar interaction random effects are still not significant

```
> anova(fm10, fm8)

Data: Multilocation
Models:
fm8: Adj ~ Trt + (1 | Location)
fm10: Adj ~ Trt + (1 | Trt:Location) + (1 | Location) + (1 | Grp)
     Df    AIC     BIC logLik deviance Chisq Chi Df Pr(>Chisq)
fm8   6 0.25442 16.347 5.8728  -11.746
fm10  8 2.35640 23.813 6.8218  -13.644 1.898      2     0.3871
```

- ▶ We have switched to ML fits because we are comparing models using `anova`. In a comparative `anova` any REML fits are refit as ML before comparison so we start with the ML fits.
- ▶ In model `fm9` the estimated variance for the scalar interaction random effects was exactly zero in the ML fit. In `fm10` the estimate is positive but still not significant.

## Vector-valued random effects

- ▶ An alternative formulation for an interaction between `Trt` and `Location` (viewed as a random-levels factor) is to use vector-valued random effects.
- ▶ We have used a similar construct in model `fm1` with vector-valued random effects (intercept and slope) for each level of `Subject`.
- ▶ One way to fit such a model is
  ```
  > fm11 <- lmer(Adj ~ Trt + (Trt | Location) + (1 |
  +     Grp), Multilocation, REML = FALSE)
  ```
  but interpretation is easier when fit as
  ```
  > fm11 <- lmer(Adj ~ Trt + (0 + Trt | Location) + (1 |
  +     Grp), Multilocation, REML = FALSE)
  ```

## Examining correlation of random effects

- ▶ The random effects summary for `fm11`
  ```
       AIC     BIC   logLik deviance
   15.8244 58.7385  8.0878 -16.1756
  Random effects:
   Groups   Name        Variance Std.Dev. Corr
   Grp      (Intercept) 0.006352 0.0797
   Location Trt1        0.119330 0.3454
            Trt2        0.093347 0.3055   0.984
            Trt3        0.104075 0.3226   0.994 0.996
            Trt4        0.099934 0.3161   0.921 0.967 0.941
   Residual             0.031647 0.1779
  Number of obs: 108, groups: Grp, 27; Location, 9
  ```
  shows very high correlations between the random effects for the levels of `Trt` within each level of `Location`.
- ▶ Such a situation may pass by unnoticed if estimates of variances and covariances are all that is reported.
- ▶ In this case (and many other similar cases) the variance-covariance matrix of the vector-valued random effects is effectively singular.

## Singular variance-covariance for random effects

- ▶ When we incorporate too many fixed-effects terms in a model we usually find out because the standard errors become very large.
- ▶ For random effects terms, especially those that are vector-valued, overparameterization is sometimes more difficult to detect.
- ▶ The REML and ML criteria for mixed-effects models seek to balance the complexity of the model versus the fidelity of the fitted values to the observed responses.
- ▶ The way "complexity" is measured in this case, a model with a singular variance-covariance matrix for the random effects is considered a good thing - it is optimally simple.
- ▶ When we have only scalar random-effects terms singularity means that one of the variance components must be exactly zero (and "near singularity" means very close to zero).

## Detecting singular random effects

- The `Lambda` slot in a `merMod` object is the triangular factor of the variance-covariance matrix.
- We can directly assess its condition number using the `kappa` (condition number) or `rcond` (reciprocal condition number) functions. Large condition numbers are bad.
- We do need to be cautious when we have a large number of levels for the grouping factors because `Lambda` will be **very** large (but also very sparse). At present the `kappa` and `rcond` functions transform the sparse matrix to a dense matrix, which could take a very long time.

```
> kappa(fm11@re@Lambda)

[1] 157833934

> rcond(fm11@re@Lambda)

[1] 4.409462e-09
```

## Using verbose model fits

- An alternative, which is recommended whenever you have doubts about a model fit, is to use `verbose=TRUE` (the lines don't wrap and we miss the interesting part here).

```
npt = 17 , n =  11
rhobeg =  0.2 , rhoend =  2e-07
   0.020:  41:     -9.00509;0.533967  1.75302 0.993757  1.29209  1.1159
  0.0020: 122:    -16.1144;0.443185  1.91331  1.66795  1.78233  1.6098
 0.00020: 234:    -16.1600;0.444143  1.93410  1.69020  1.80295  1.6311
 2.0e-05: 482:    -16.1756;0.447997  1.94216  1.69082  1.80318  1.6362
 2.0e-06: 530:    -16.1756;0.447999  1.94183  1.69048  1.80285  1.6360
 2.0e-07: 570:    -16.1756;0.447998  1.94182  1.69047  1.80284  1.6360
At return
616:    -16.175574: 0.447997  1.94182  1.69047  1.80284  1.63601 0.3032
> fm11@re@theta

[1]   4.479973e-01  1.941823e+00  1.690472e+00   1.802841e+00
[5]   1.636013e+00  3.032622e-01  1.790633e-01   6.128860e-01
[9]   7.972120e-02 -3.249879e-01  1.380652e-07
```

## What to watch for in the verbose output

- In this model the criterion is being optimized with respect to 11 parameters.
- These are the variance component parameters, $\boldsymbol{\theta}$. The fixed-effects coefficients, $\boldsymbol{\beta}$, and the common scale parameter, $\sigma$, are at their conditionally optimal values.
- Generally it is more difficult to estimate a variance parameter (either a variance or a covariance) than it is to estimate a coefficient. Estimating 11 such parameters requires a considerable amount of information.
- Some of these parameters are required to be non-negative. When they become zero or close to zero ($2.7 \times 10^{-7}$, in this case) the variance-covariance matrix is degenerate.
- The `@re@lower` slot contains the lower bounds. Parameter components for which `@re@lower` is `-Inf` are unbounded. The ones to check are those for which `@re@lower` is `0`.

## Another example of singular variance-covariance

- The `Early` data in the `mlmRev` package are from a study on early childhood cognitive development as influenced by a treatment. These data are discussed in **Applied Longitudinal Data Analysis** (2003) by Singer and Willett.
- A model with random effects for slope and intercept is

```
> Early <- within(Early, tos <- age - 0.5)
> fm12 <- lmer(cog ~ tos + trt:tos + (tos | id), Early,
+     verbose = TRUE)

npt = 7 , n =  3
rhobeg =  0.2 , rhoend =  2e-07
   0.020:  11:     2368.50; 1.09296 -0.173139 0.0953204
  0.0020:  30:     2364.50; 1.48770 -0.374305 0.0138819
 0.00020:  42:     2364.50; 1.48462 -0.372458 0.00762182
 2.0e-05:  58:     2364.50; 1.48417 -0.372319 0.00114304
 2.0e-06:  74:     2364.50; 1.48420 -0.372480  0.00000
 2.0e-07:  80:     2364.50; 1.48420 -0.372481  0.00000
At return
 84:     2364.5016:  1.48420 -0.372481  0.00000
```

## Fitted model for the Early data

```
Linear mixed model fit by REML ['merMod']
Formula: cog ~ tos + trt:tos + (tos | id)
   Data: Early
REML criterion at convergence: 2364.502
Random effects:
 Groups   Name        Variance Std.Dev. Corr
 id       (Intercept) 166.40   12.900
          tos          10.48    3.237   -1.000
 Residual              75.54    8.691
Number of obs: 309, groups: id, 103

Fixed effects:
            Estimate Std. Error t value
(Intercept)  120.783      1.824   66.22
tos          -22.470      1.494  -15.04
tos:trtY       7.646      1.447    5.28
```
Here is it obvious that there is a problem. However, Singer and
Willett did not detect this in model fits from SAS PROC MIXED
or MLWin, which reported a covariance estimate.

## Fixed-effects or random-effects?

- ▶ Earlier we described the distinction between fixed and random
  effects as dependent on the repeatability of the levels.
- ▶ This is the basis for the distinction but the number of levels
  observed must also be considered.
- ▶ Fitting mixed-effects models requires data from several levels
  of the grouping factor. Even when a factor represents a
  random selection (say sample transects in an ecological study)
  it is not practical to estimate a variance component from only
  two or three observed levels.
- ▶ At the other extreme, a census of a large number of levels can
  be modeled with random effects even though the observed
  levels are not a sample.

## Other practical issues

- ▶ In some disciplines there is an expectation that data will be
  analyzed starting with the most complex model and
  evaluating terms according to their p-values.
- ▶ This can be appropriate for carefully balanced, designed
  experiments. It is rarely a good approach on observational,
  imbalanced data.
- ▶ Bear in mind that this approach was formulated when
  graphical and computational capabilities were very limited.
- ▶ A more appropriate modern approach is to explore the data
  graphically and to fit models sequentially, comparing these
  fitted models with tests such as the LRT.